

Quotient-Space Diffusion Models

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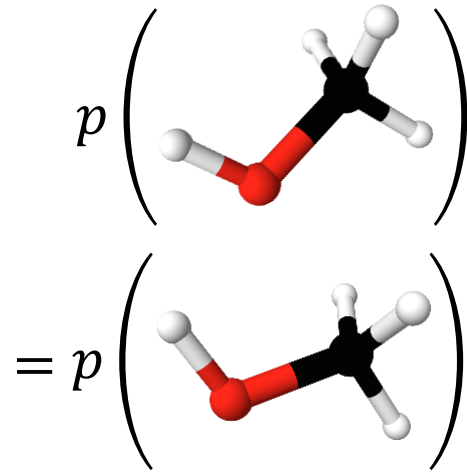
Di He^{*}



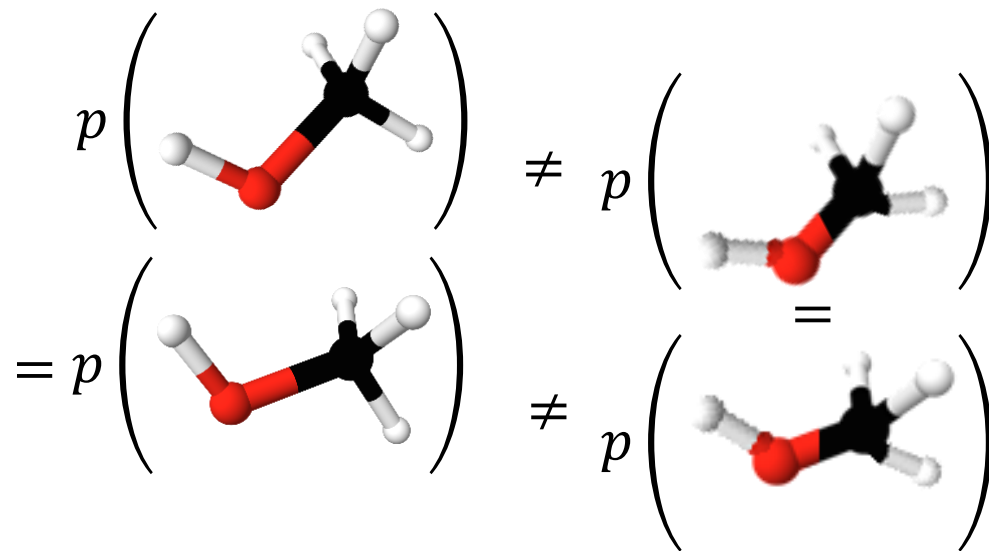
Chang Liu^{*}



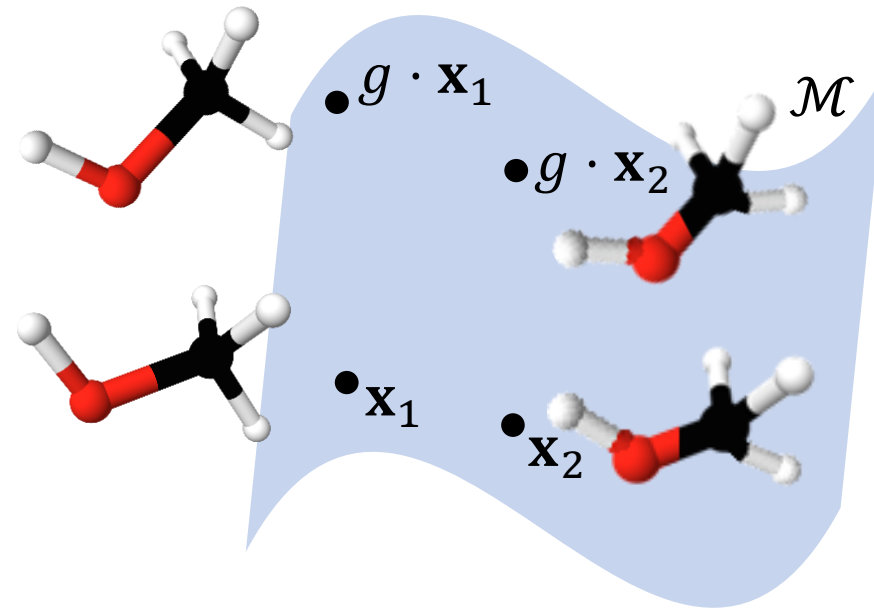
Symmetry under Group (\mathcal{G}) Actions



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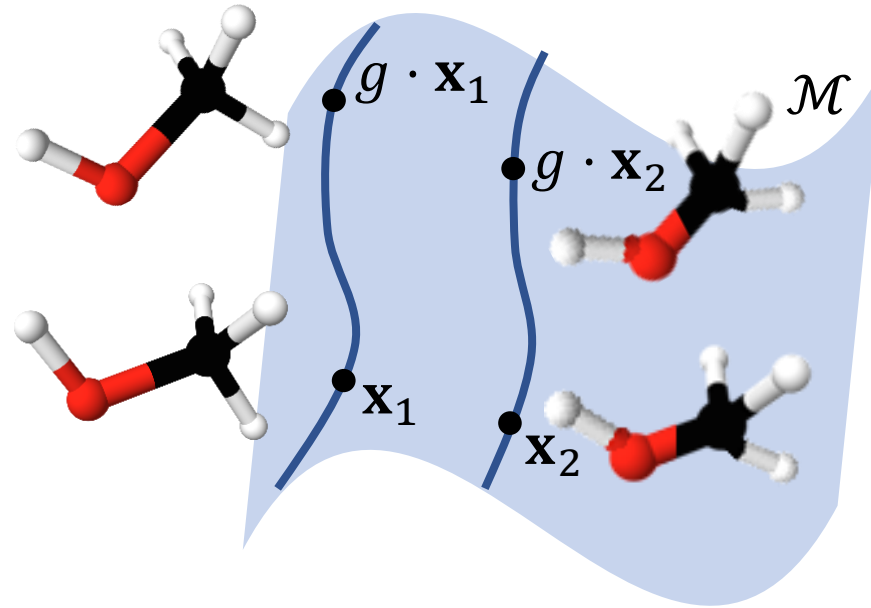


Symmetry under Group (\mathcal{G}) Actions

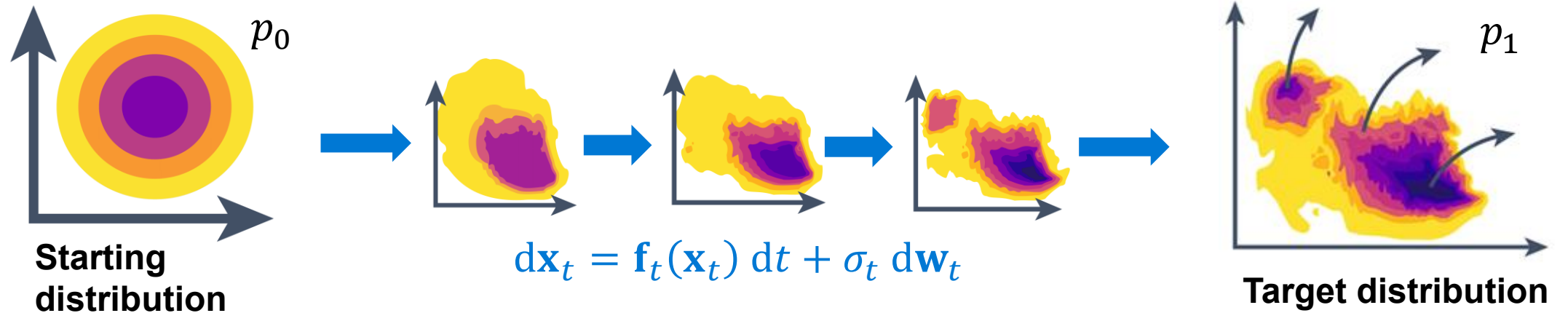


Symmetry under Group (\mathcal{G}) Actions

$$\mathcal{G} = \text{SE}(3)$$



Equivariant Diffusion Models



Generation:

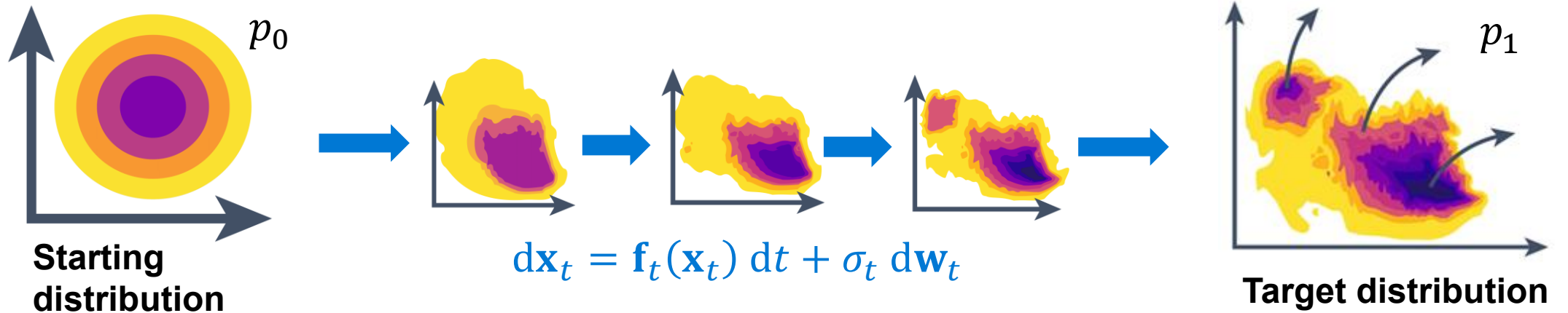
- Invariant start $p_0(\mathbf{x}_0) = p_0(g \cdot \mathbf{x}_0)$
- Equivariant process¹ $(g \cdot \cdot)_* \mathbf{f}_t(\mathbf{x}_t) = \mathbf{f}_t(g \cdot \mathbf{x}_t)$



- ✓ Invariant target $p_1(\mathbf{x}_1) = p_1(g \cdot \mathbf{x}_1)$

¹: w_t is invariant if the group action is isometric.

Equivariant Diffusion Models



Generation:

- Invariant start $p_0(\mathbf{x}_0) = p_0(g \cdot \mathbf{x}_0)$
- Equivariant process¹ $(g \cdot \cdot)_* \mathbf{f}_t(\mathbf{x}_t) = \mathbf{f}_t(g \cdot \mathbf{x}_t)$



- ✓ Invariant target $p_1(\mathbf{x}_1) = p_1(g \cdot \mathbf{x}_1)$

Training:

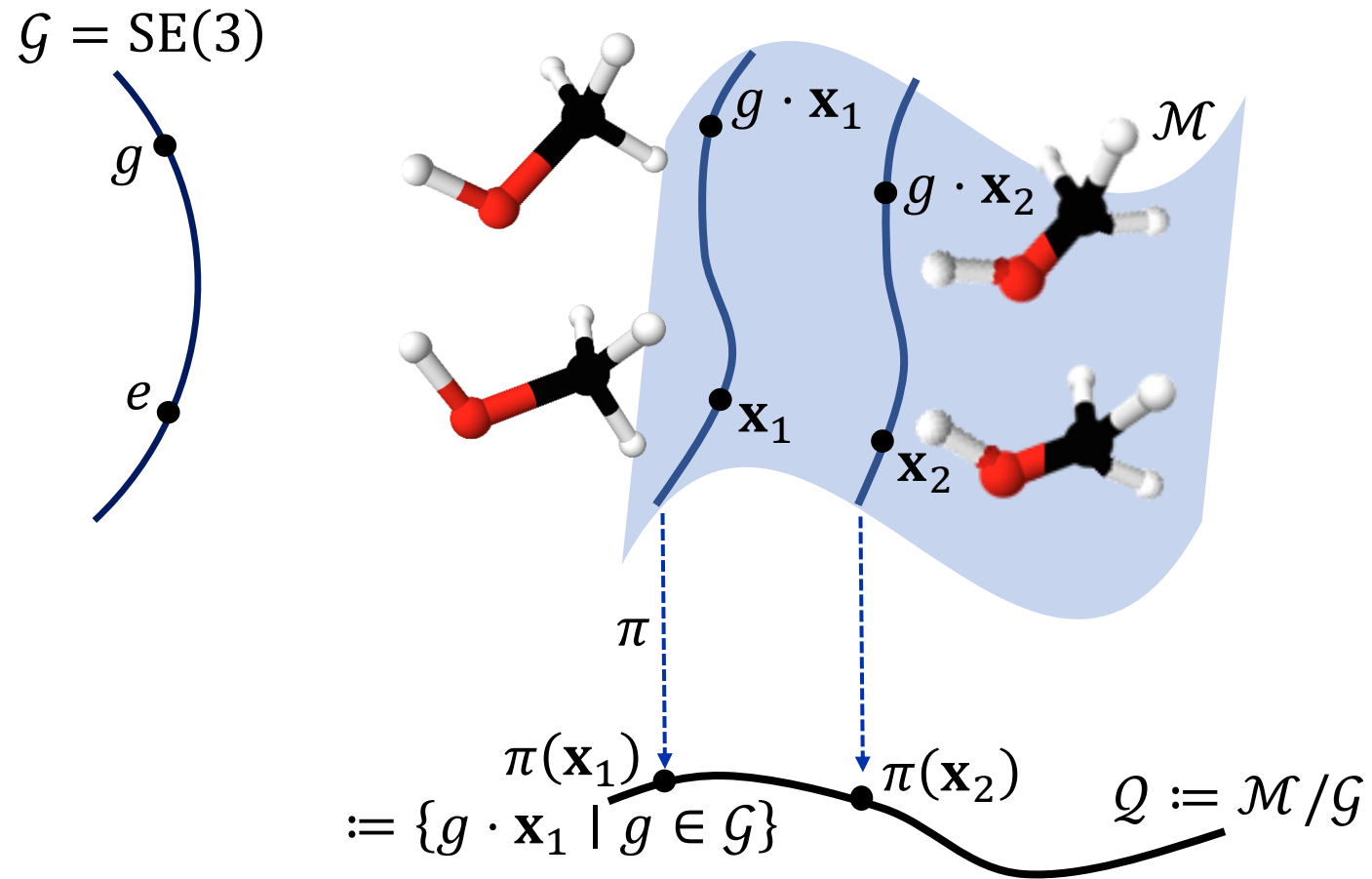
$$\mathbf{f}_{\theta,t}(\mathbf{x}_t) = \text{coeff1}_t \mathbf{x}_t + \text{coeff2}_t \mathbf{D}_{\theta,t}(\mathbf{x}_t),$$

where $\min_{\theta} \mathbb{E}_{p(\mathbf{x}_1, \mathbf{x}_t)} \|\mathbf{D}_{\theta,t}(\mathbf{x}_t) - \mathbf{x}_1\|^2$.

- ? $\mathbf{D}_{\theta,t}(\mathbf{x}_t)$ still learns a full mapping:
Can we leverage *symmetry* to **reduce learning difficulty**?

¹: \mathbf{w}_t is invariant if the group action is isometric.

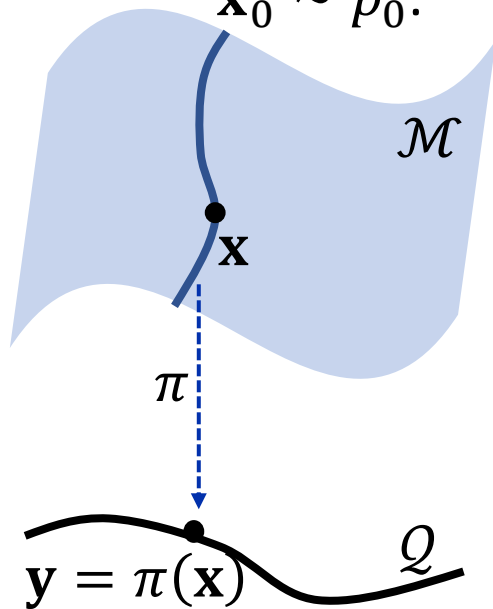
The View under Quotient Space



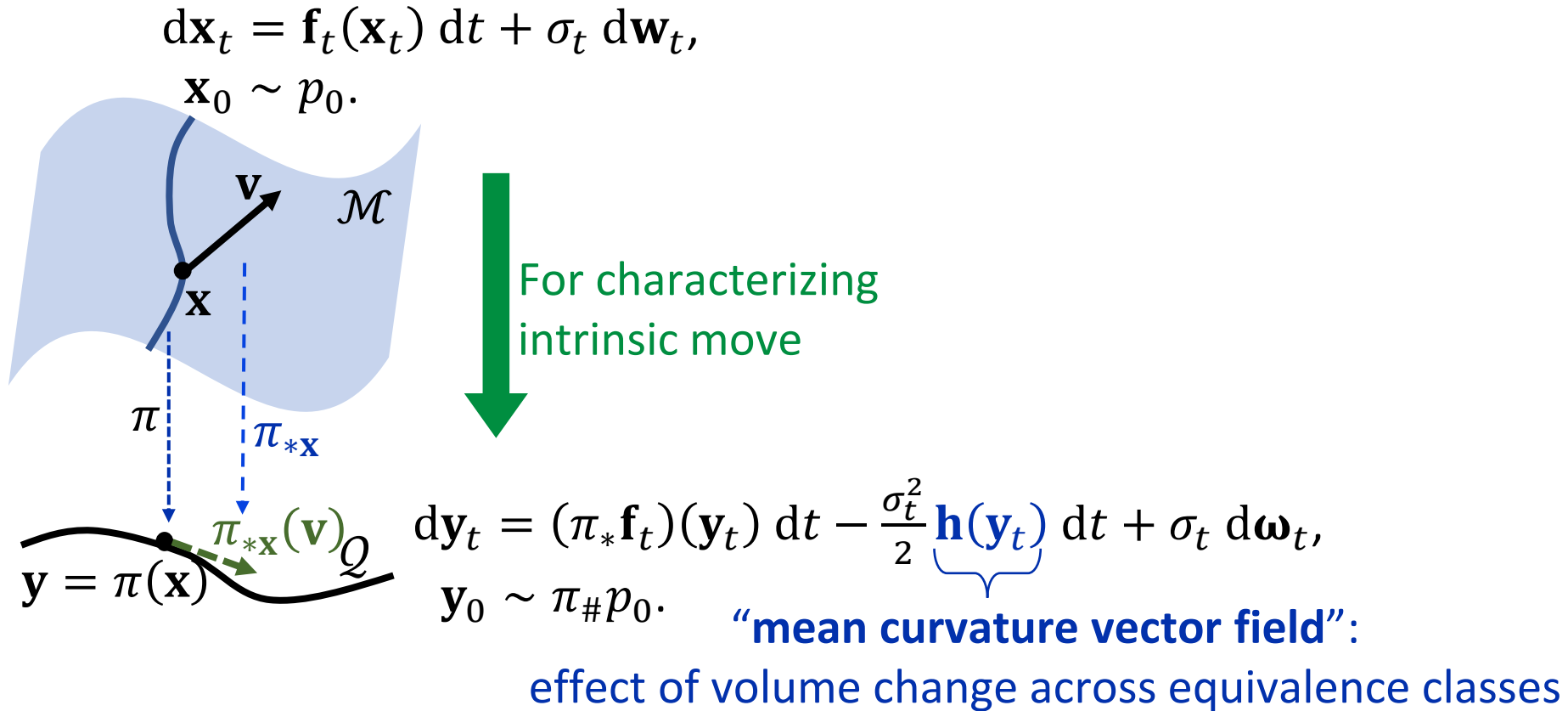
Quotient-Space Diffusion Process

$$d\mathbf{x}_t = \mathbf{f}_t(\mathbf{x}_t) dt + \sigma_t d\mathbf{w}_t,$$

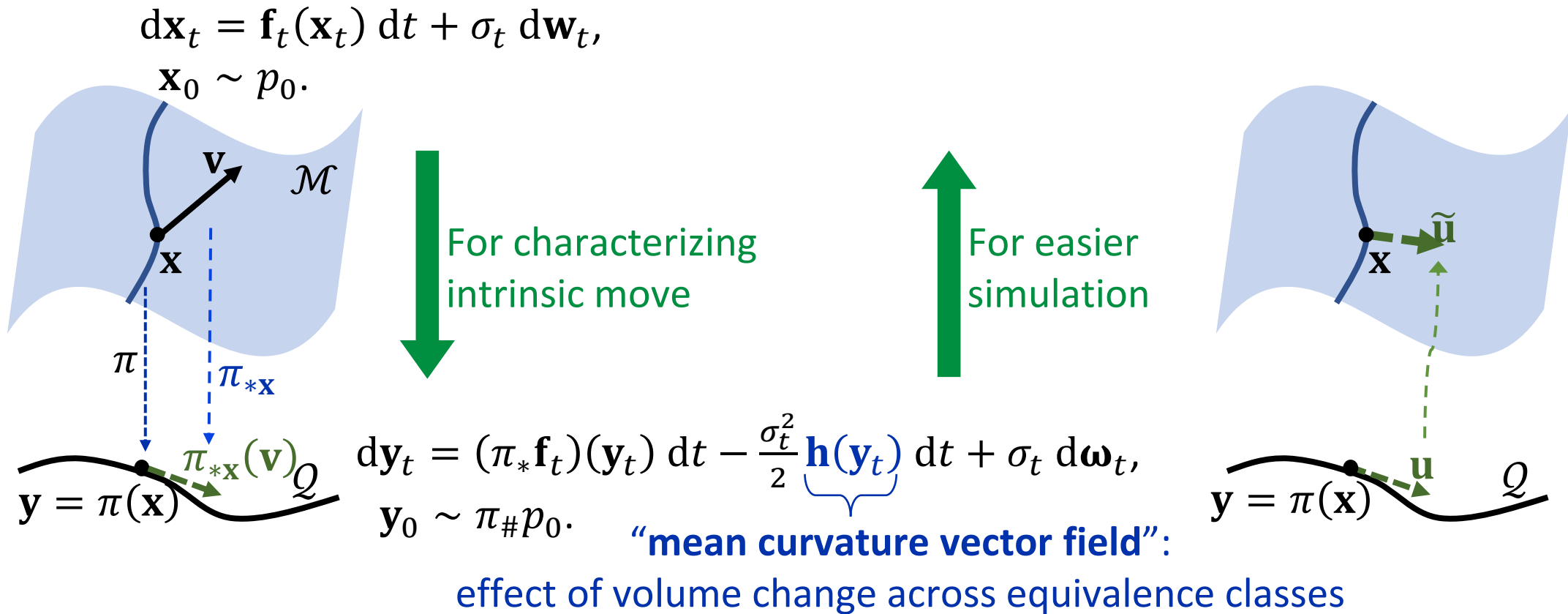
$$\mathbf{x}_0 \sim p_0.$$



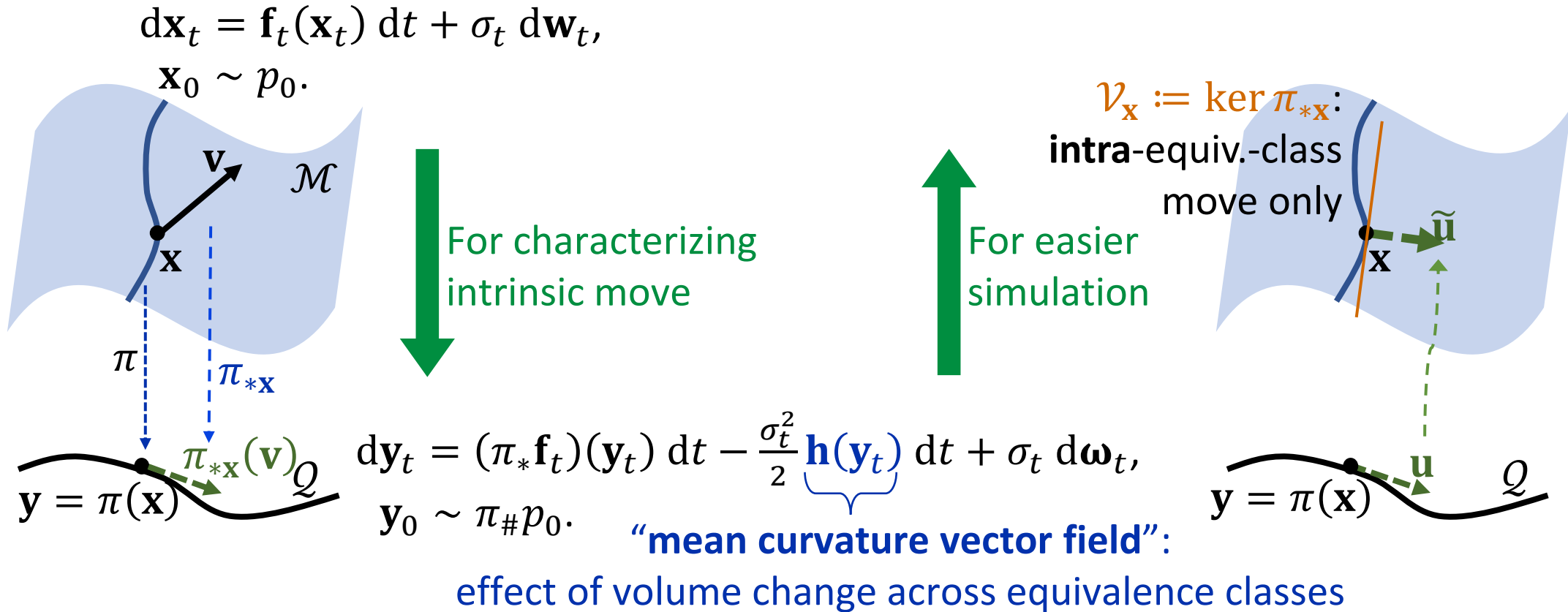
Quotient-Space Diffusion Process



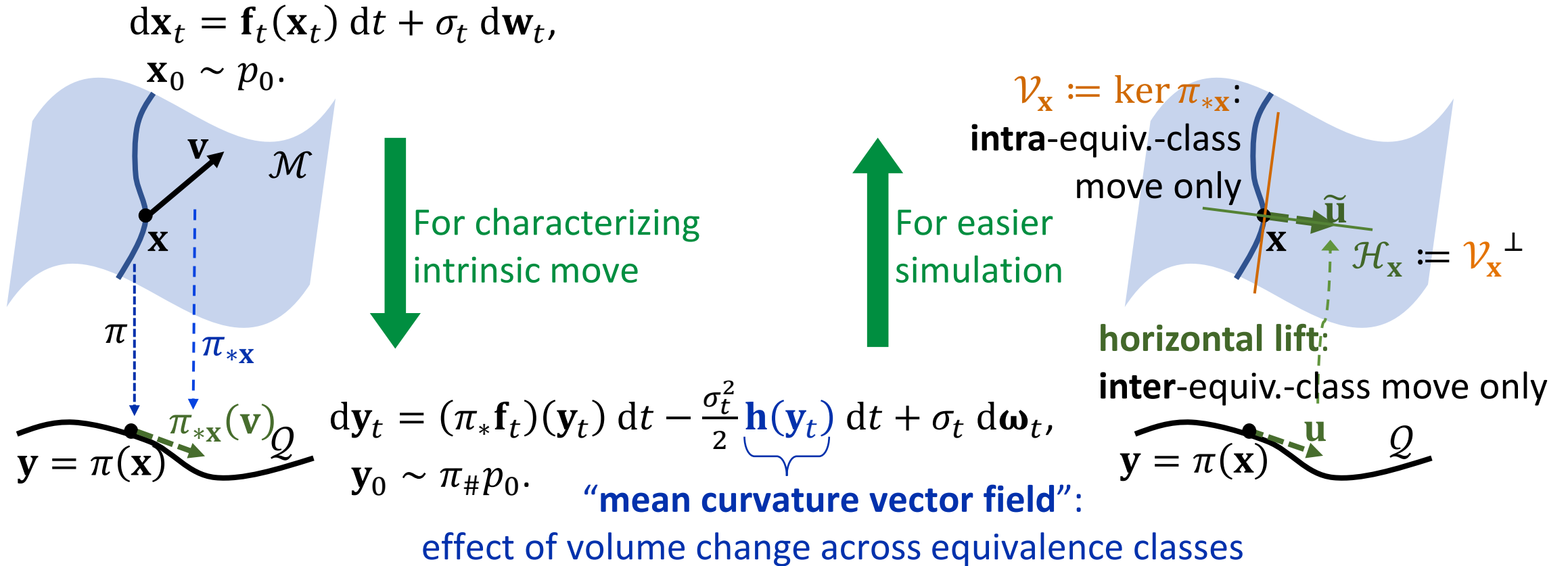
Quotient-Space Diffusion Process



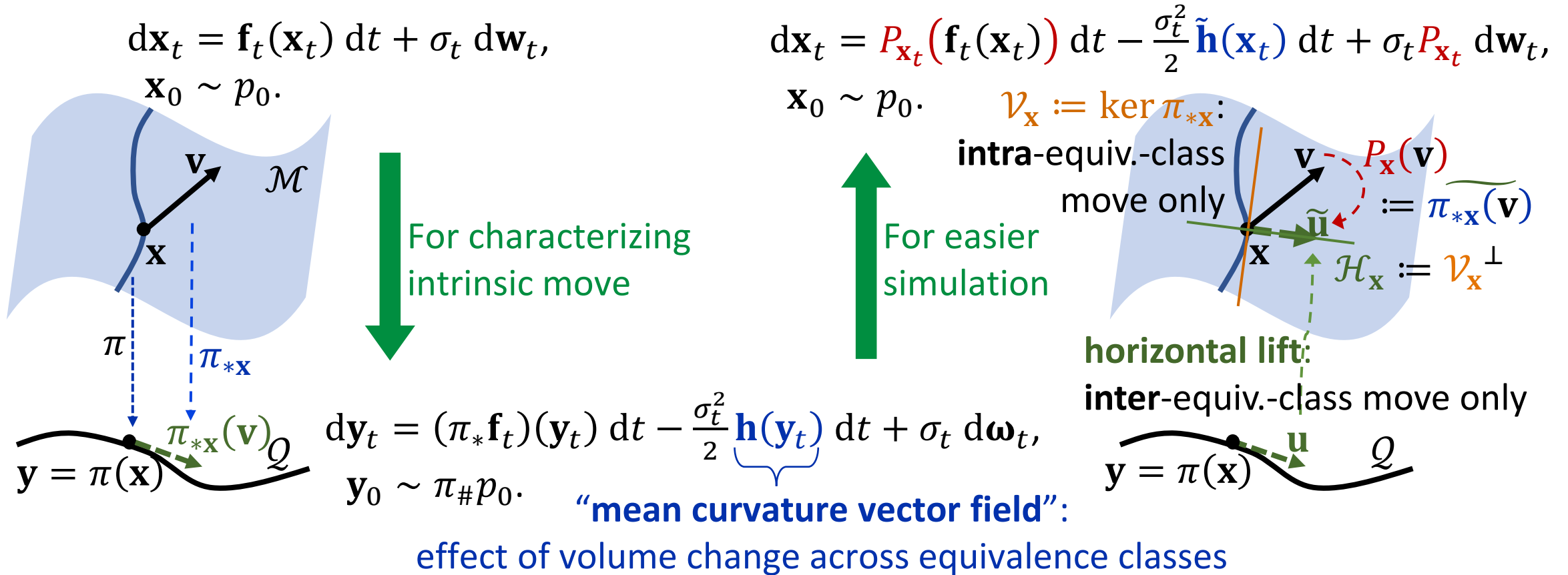
Quotient-Space Diffusion Process



Quotient-Space Diffusion Process



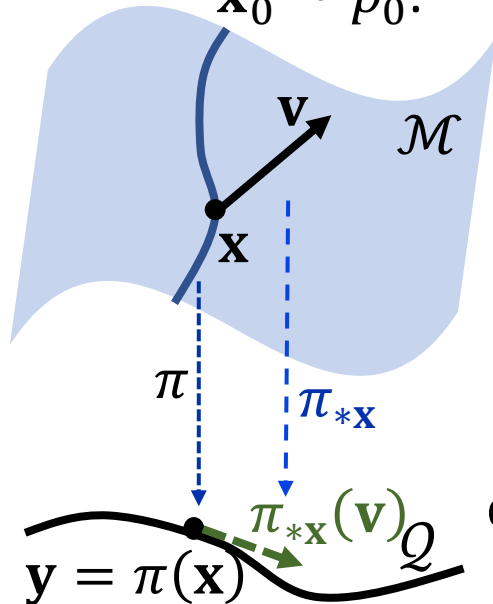
Quotient-Space Diffusion Process



Quotient-Space Diffusion Process

$$d\mathbf{x}_t = \mathbf{f}_t(\mathbf{x}_t) dt + \sigma_t d\mathbf{w}_t,$$

$$\mathbf{x}_0 \sim p_0.$$



For characterizing
intrinsic move

$$d\mathbf{y}_t = (\pi_*\mathbf{f}_t)(\mathbf{y}_t) dt - \frac{\sigma_t^2}{2} \mathbf{h}(\mathbf{y}_t) dt + \sigma_t d\mathbf{w}_t$$

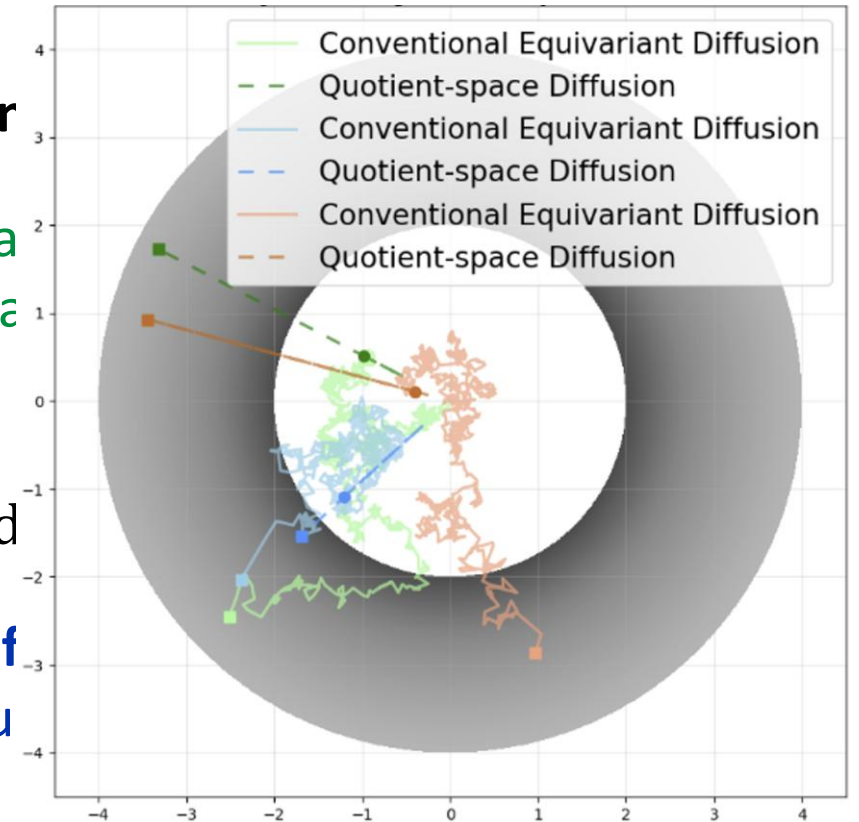
$$\mathbf{y}_0 \sim \pi_{\#}p_0.$$

“mean curvature vector \mathbf{f}
effect of volume change across equ

$$d\mathbf{x}_t = P_{\mathbf{x}_t}(\mathbf{f}_t(\mathbf{x}_t)) dt - \frac{\sigma_t^2}{2} \tilde{\mathbf{h}}(\mathbf{x}_t) dt + \sigma_t P_{\mathbf{x}_t} d\mathbf{w}_t,$$

$$\mathbf{x}_0 \sim p_0.$$

For ea
simula



Quotient-Space Diffusion Process

$$d\mathbf{x}_t = \mathbf{f}_t(\mathbf{x}_t) dt + \sigma_t d\mathbf{w}_t,$$

$$\mathbf{x}_0 \sim p_0.$$

$$d\mathbf{x}_t = P_{\mathbf{x}_t}(\mathbf{f}_t(\mathbf{x}_t)) dt - \frac{\sigma_t^2}{2} \tilde{\mathbf{h}}(\mathbf{x}_t) dt + \sigma_t P_{\mathbf{x}_t} d\mathbf{w}_t,$$

$$\mathbf{x}_0 \sim p_0.$$

Shape space: $Q = \mathbb{R}_{\text{CoM}}^{3N}/\text{SO}(3)$:

for $\mathbf{x} = [\vec{\mathbf{x}}^{(n)}]_{n'}$, $\mathbf{v} = [\vec{\mathbf{v}}^{(n)}]_{n'}$,

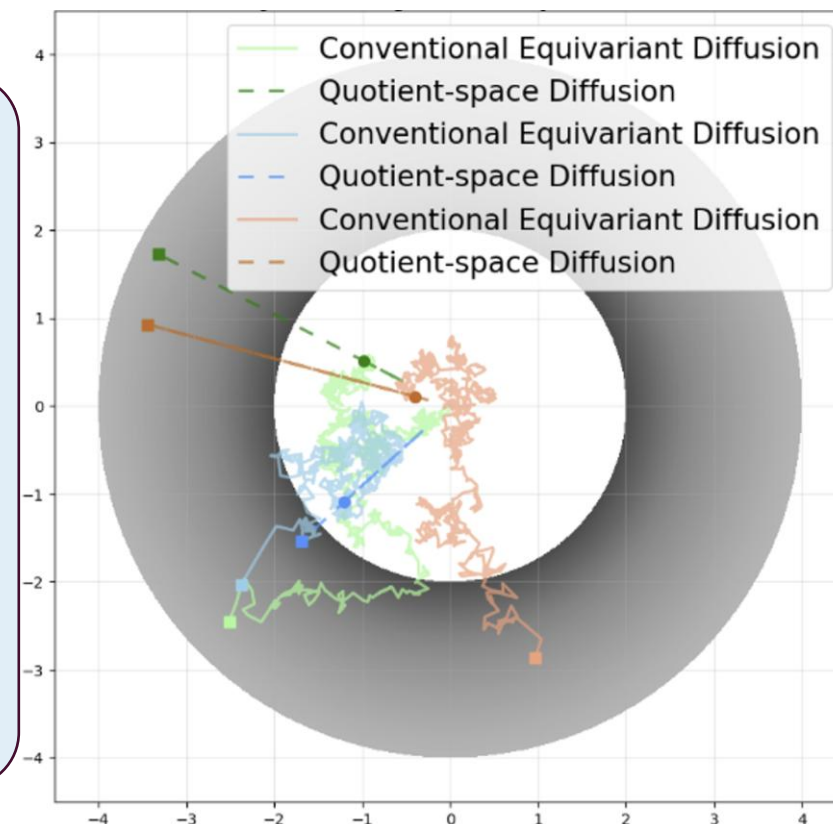
$$P_{\mathbf{x}}(\mathbf{v}) = \left[\vec{\mathbf{v}}^{(n)} - \mathbf{K}(\mathbf{x})^{-1} \left(\sum_{n'=1}^N \vec{\mathbf{x}}^{(n')} \times \vec{\mathbf{v}}^{(n')} \right) \times \vec{\mathbf{x}}^{(n)} \right]_{n'},$$

$$\tilde{\mathbf{h}}(\mathbf{x}) = \left[(\mathbf{K}(\mathbf{x})^{-1} - \text{tr}(\mathbf{K}(\mathbf{x})^{-1}) \mathbf{I}) \vec{\mathbf{x}}^{(n)} \right]_{n'},$$

where $\mathbf{K}(\mathbf{x}) := \sum_{n=1}^N \|\vec{\mathbf{x}}^{(n)}\|^2 \mathbf{I} - \sum_{n=1}^N \vec{\mathbf{x}}^{(n)} \vec{\mathbf{x}}^{(n)\top} \in \mathbb{R}^{3 \times 3}$.

Removes total/rigid-body angular momentum!

Leaves only deformation (change of shape)!



Reduction of Learning Difficulty

Training convenience: $\mathbb{E}\|\mathbf{D}_{\theta,t}(\mathbf{x}_t) - \mathbf{x}_1\|^2 \rightarrow \mathbb{E}\|P_{\mathbf{x}_t}(\mathbf{D}_{\theta,t}(\mathbf{x}_t) - \mathbf{x}_1)\|^2$:
 no need to learn **anything** corresponding to **intra-equiv.-class** movement!

Training strategy for \mathbf{D}_θ	Optimal solution of \mathbf{D}_θ	Reduction of learning difficulty		Sampling compatibility
		Removal of equivalent DoFs	Removal of variance on equivalent DoFs	
Conventional loss $\mathbb{E}\ \mathbf{D}_\theta(\mathbf{x}_t, t) - \mathbf{x}_1\ ^2$	$\mathbb{E}[\mathbf{x}_1 \mathbf{x}_t]$	x	x	✓
quotient-space diffusion loss $\mathbb{E}\ P_{\mathbf{x}_t}(\mathbf{D}_\theta(\mathbf{x}_t, t) - \mathbf{x}_1)\ ^2$	$\mathbb{E}[P_{\mathbf{x}_t}(\mathbf{x}_1) \mathbf{x}_t] + \mathbf{v}^\nu$ for arbitrary $\mathbf{v}^\nu \in \text{Ker}(P_{\mathbf{x}_t})$	✓	✓	✓

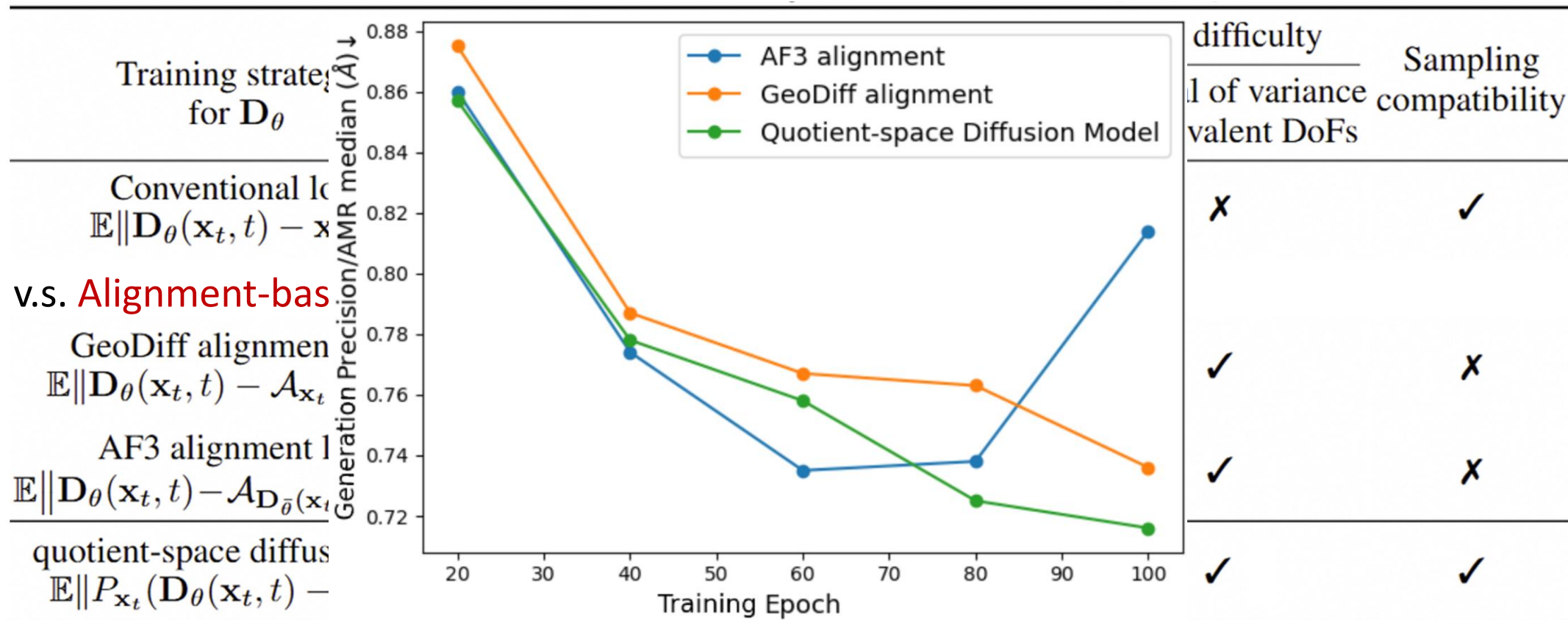
Reduction of Learning Difficulty

Training convenience: $\mathbb{E} \|\mathbf{D}_{\theta,t}(\mathbf{x}_t) - \mathbf{x}_1\|^2 \rightarrow \mathbb{E} \|P_{\mathbf{x}_t}(\mathbf{D}_{\theta,t}(\mathbf{x}_t) - \mathbf{x}_1)\|^2$:
 no need to learn **anything** corresponding to **intra-equiv.-class** movement!

Training strategy for \mathbf{D}_θ	Optimal solution of \mathbf{D}_θ	Reduction of learning difficulty		Sampling compatibility
		Removal of equivalent DoFs	Removal of variance on equivalent DoFs	
Conventional loss $\mathbb{E} \ \mathbf{D}_\theta(\mathbf{x}_t, t) - \mathbf{x}_1\ ^2$	$\mathbb{E}[\mathbf{x}_1 \mathbf{x}_t]$	✗	✗	✓
v.s. Alignment-based methods: $\mathcal{A}_y(\mathbf{x}) := \operatorname{argmin}_{\mathbf{x}' \in \{g \cdot \mathbf{x} g \in \mathcal{G}\}} d(\mathbf{x}', \mathbf{y})$				
GeoDiff alignment loss $\mathbb{E} \ \mathbf{D}_\theta(\mathbf{x}_t, t) - \mathcal{A}_{\mathbf{x}_t}(\mathbf{x}_1)\ ^2$	$\mathbb{E}[\mathcal{A}_{\mathbf{x}_t}(\mathbf{x}_1) \mathbf{x}_t]$	✗	✓	✗
AF3 alignment loss $\mathbb{E} \ \mathbf{D}_\theta(\mathbf{x}_t, t) - \mathcal{A}_{\mathbf{D}_{\bar{\theta}}(\mathbf{x}_t, t)}(\mathbf{x}_1)\ ^2$	$g \cdot \mathbb{E}[\mathcal{A}_{\mathbf{x}_t}(\mathbf{x}_1) \mathbf{x}_t]$ for arbitrary $g \in \mathcal{G}$	✓	✓	✗
quotient-space diffusion loss $\mathbb{E} \ P_{\mathbf{x}_t}(\mathbf{D}_\theta(\mathbf{x}_t, t) - \mathbf{x}_1)\ ^2$	$\mathbb{E}[P_{\mathbf{x}_t}(\mathbf{x}_1) \mathbf{x}_t] + \mathbf{v}^\mathcal{V}$ for arbitrary $\mathbf{v}^\mathcal{V} \in \operatorname{Ker}(P_{\mathbf{x}_t})$	✓	✓	✓

Reduction of Learning Difficulty

Training convenience: $\mathbb{E} \|\mathbf{D}_{\theta,t}(\mathbf{x}_t) - \mathbf{x}_1\|^2 \rightarrow \mathbb{E} \|P_{\mathbf{x}_t}(\mathbf{D}_{\theta,t}(\mathbf{x}_t) - \mathbf{x}_1)\|^2$:
 no need to learn **anything** corresponding to **intra-equiv.-class movement!**



Results: Molecular Structure Generation

Datasets	Methods	Recall				Precision			
		Coverage \uparrow		AMR \downarrow		Coverage \uparrow		AMR \downarrow	
		mean	median	mean	median	mean	median	mean	median
GEOM-QM9 (Positive samples are within 0.5 Å RMSD.)	CGCF	69.47	96.15	0.425	0.374	38.20	33.33	0.711	0.695
	GeoDiff	76.50	100.00	0.297	0.229	50.00	33.50	1.524	0.510
	GeoMol	91.50	100.00	0.225	0.193	87.60	100.00	0.270	0.241
	Torsional Diff.	92.80	100.00	0.178	0.147	92.70	100.00	0.221	0.195
	MCF	95.0	100.00	0.103	0.044	93.7	100.00	0.119	0.055
	ET-Flow(SO(3))	95.98	100.00	0.076	0.030	92.10	100.00	0.110	0.047
	+ Geodiff alignment	95.71	100.00	0.085	0.040	95.20	100.00	0.098	0.050
	+ AF3 alignment	92.67	100.00	0.131	0.070	84.38	100.00	0.205	0.146
	+ Quotient-space diffusion	96.40	100.00	0.069	0.024	93.30	100.00	0.096	0.036
	GEOM-DRUGS (Positive samples are within 0.75 Å RMSD.)	GeoDiff	42.10	37.80	0.835	0.809	24.90	14.50	1.136
GeoMol		44.60	41.40	0.875	0.834	43.00	36.40	0.928	0.841
Torsional Diff.		72.70	80.00	0.582	0.565	55.20	56.90	0.778	0.729
MCF - S (13M)		79.4	87.5	0.512	0.492	57.4	57.6	0.761	0.715
MCF - B (62M)		84.0	91.5	0.427	0.402	64.0	66.2	0.667	0.605
MCF - L (242M)		84.7	92.2	0.390	0.247	66.8	71.3	0.618	0.530
ET-Flow (8.3M)		79.53	84.57	0.452	0.419	74.38	81.04	0.541	0.470
+ reproduction		78.94	84.24	0.489	0.472	66.24	70.42	0.651	0.595
+ Quotient-space diffusion		<u>79.86</u>	<u>85.71</u>	<u>0.459</u>	<u>0.433</u>	72.70	79.63	0.565	0.501
ET-Flow(SO(3)) (9.1M)		78.18	83.33	0.480	0.459	67.27	71.15	0.637	0.567
+ reproduction		74.91	80.90	0.541	0.515	60.33	62.71	0.724	0.665
+ Geodiff alignment		75.11	80.74	0.545	0.526	59.58	60.48	0.734	0.678
+ AF3 alignment		71.66	76.09	0.572	0.570	52.21	50.00	0.828	0.793
+ Quotient-space diffusion		<u>78.50</u>	<u>84.20</u>	<u>0.477</u>	<u>0.455</u>	<u>67.35</u>	<u>71.42</u>	<u>0.635</u>	<u>0.563</u>

Results: Protein Structure Generation

Settings	Methods	Designability (%) \uparrow	FPSD vs.		fS	fJSD vs.	
			PDB \downarrow	AFDB \downarrow	(C/A/T) \uparrow	PDB \downarrow	AFDB \downarrow
Representative References	FrameDiff	65.4	194.2	258.1	2.46/5.78/23.35	1.04	1.42
	FoldFlow (base)	96.6	601.5	566.2	1.06/1.79/9.72	3.18	3.10
	FoldFlow (stoc.)	97.0	543.6	520.4	1.21/2.09/11.59	3.69	2.71
	FoldFlow (OT)	97.2	431.4	414.1	1.35/3.10/13.62	2.90	2.32
	FrameFlow	88.6	129.9	159.9	2.52/5.88/27.00	0.68	0.91
	ESM3	22.0	933.9	855.4	3.19/6.71/17.73	1.53	0.98
	Chroma	74.8	189.0	184.1	2.34/4.95/18.15	1.00	1.08
	RFDiffusion	94.4	253.7	252.4	2.25/5.06/19.83	1.21	1.13
	Proteus	94.2	225.7	226.2	2.26/5.46/16.22	1.41	1.37
Genie2	95.2	350.0	313.8	1.55/3.66/11.65	2.21	1.70	
SDE Sampling	Proteína $\mathcal{M}_{\text{FS}}^{\text{small}}, \gamma = 0.35$	96.0	386.5	378.2	1.77/4.97/17.78	2.17	1.73
	+ Quotient-space diffusion	97.6	274.7	277.1	2.24/6.69/20.99	1.68	1.55
	Proteína $\mathcal{M}_{\text{FS}}^{\text{small}}, \gamma = 0.45$	92.2	332.9	320.4	1.83/5.01/20.22	1.93	1.49
	+ Quotient-space diffusion	92.6	244.5	246.3	2.24/6.68/23.47	1.43	1.28
ODE Sampling	Proteína \mathcal{M}_{FS}	19.6	85.4	21.4	2.51/5.65/27.35	0.59	0.09
	Proteína $\mathcal{M}_{\text{FS}}^{\text{small}}$	13.8	83.2	21.9	2.45/5.63/31.76	0.58	0.12
	+ AF3 alignment	3.8	229.0	82.4	2.18/4.30/14.28	1.35	0.36
	+ Quotient-space diffusion	15.6	69.9	17.6	2.57/6.40/32.14	0.41	0.11

Thanks!

Check out our paper

